4^{th} iso thm. Let $N \trianglelefteq G$ and $\pi_N : G \to G/N$ be the projection map.

Then (1) $\Pi : \{L \mid N \trianglelefteq L < G\} \to \{\bar{L} < G/N\}$, $L \mapsto \pi_N(L)\}$ is a bijection.

(2) Π gives a bijection between the normal subgps $L \trianglelefteq G$ with $N < L$ with the normal subgps $\bar{L}$ of G/N.

Pf. (1) For any $\bar{L} < G/N$, $\pi_N^{-1}(\bar{L})$ is a subgp of G that contains N. Moreover $L \mapsto \pi_N(L)$, $\bar{L} \mapsto \pi_N^{-1}(\bar{L})$ are inverse to each other.

(2) $gL = Lg$ in G $\Rightarrow$ $(gN)L = L(gN)$ in G/N.

So L normal $\Rightarrow$ $\bar{L}$ normal.

If $\bar{L}$ is normal, then $\pi_N(gLg^{-1}) = \pi_N(g)\pi_N(L)\pi_N(g)^{-1}$

$$= \bar{L}$$

So $gLg^{-1} \subseteq \pi_N^{-1}(\bar{L}) = L$ So. L is normal $\square$.

Def. A gp G is called <u>simple</u> if $G \neq \{1\}$ and has no proper nontrivial normal subgps.

Examples: • Any simple abelian gp must be cyclic of prime order.

• A_n is simple for $n \geq 5$

Def. A normal subgp $M \underset{\neq}{\trianglelefteq} G$ is <u>maximal</u> if $\nexists\, N \trianglelefteq G$ s.t. $M \subsetneq N \subsetneq G$.

Prop. $M \trianglelefteq G$ is max iff G/M is simple.

Pf. By 4th iso thm. $\square$

Rmk. Simple gps are the "building blocks" of gps.

Series of gps.

Def. Let $\{1\} = H_0 < H_1 < \cdots < H_n = G$ be a finite chain of subgps.
We say that it is subnormal if $H_i \trianglelefteq H_{i+1}$ $\forall i$
normal if $H_i \trianglelefteq G$ $\forall i$.

The quotient gps H_{i+1}/H_i are called the quotient (or factor) gps of the series.

Def. A subnormal (resp. normal) series $\{H_i\}$ is a composition (resp. principal) series if all its quotient gp H_{i+1}/H_i are simple.
Here $\{H_{i+1}/H_i\}$ are now called composition factors.

Rmk: Every finite gp has a composition series. / $\mathbb{Z}$ does not have.

- Given a composition series
$$\{1\} = H_0 < H_1 < \cdots < H_n = G.$$
We have a sequence of short exact sequences

$$1 \to \underline{H_1} \to H_2 \to \underline{H_2/H_1} \to 1$$

finite simple gps

$$1 \to H_2 \to H_3 \to \underline{H_3/H_2} \to 1$$

$$\vdots$$

$$1 \to H_{n-1} \to H_n = G \to \underline{H_n/H_{n-1}} \to 1$$

Thm (Jordan–Hölder) Let G be a finite gp. If

$$\{1\} = H_0 < H_1 < \cdots < H_n = G$$
$$\{1\} = K_0 < K_1 < \cdots < K_m = G.$$

are two composition series for G. Then $m=n$ and $\exists\, \sigma \in S_n$, s.t.

$$H_{i+1}/H_i \cong K_{\sigma(i)+1}/K_{\sigma(i)}$$

Pf. Induction on $|G|$.

Case 1: $H_{n-1} = K_{m-1}$. In this case, follows from induction on H_{n-1}

Case 2: $H_{n-1} \neq K_{m-1}$. In this case $H_{n-1}, K_{m-1} \lhd G$. maximal.

So $H_{n-1}K_{m-1} = G$.

So $G/H_{n-1} \cong K_{m-1}/H_{n-1}\cap K_{m-1}$, $G/K_{m-1} \cong H_{n-1}/H_{n-1}\cap K_{m-1}$.

Let $J = H_{n-1}\cap K_{m-1}$. Then J is a max normal subgp of both H_{n-1} and K_{m-1}. Now by inductive hypothesis on H_{n-1}, J and K_{m-1}, we have

① $\{1\} = H_0 < \cdots < H_{n-2} < H_{n-1} < H_n = G$

② $\{1\} = H_0 < \cdots < J < H_{n-1} < H_n = G.$

③ $\{1\} = K_0 < \cdots < J < K_{m-1} < K_m = G$

④ $\{1\} = K_0 < \cdots < K_{m-2} < K_{m-1} < K_m = G.$

} use a compositi series of J.

By inductive hypothesis the composition factors of ① and ② are the same, the composition factors of ③ and ④ are the same.

Also $H_n / H_{n-1} \cong K_{m-1}/J$ and $K_m / K_{m-1} \cong H_{n-1}/J$.

So the composition factors of ② and ③ are the same $\square$

Rmk. Apply the theorem to finite abelian gps, we see that the prime factorization of a given positive integer is unique

order of compos factors   order of G.

Hölder program: every finite gp is built from finite simple gps

① Classify all finite simple gps ← completed in 2004

② classify all possible ways of building a gp ← unknown from given finite simple gps

Def. A gp G is called solvable if it has a subnormal series whose quotient gps are all abelian

In other words, a solvable gp is gp built from (successive extensions of) abelian gp.

Examples. • All abelian gps are solvable

• S_3 is solvable   Ex S_4 is solvable

• S_n is not solvable when $n \geq 5$.

• $B := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq GL_2(F)$ is solvable

Since $U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ is normal in B and $B/U \cong F^\times \times F^\times$

Prop. Every subgp and quotient gp of a solvable gp is solvable

Pf. If G is solvable with solvable series

$$\{1\} = K_0 < K_1 < \cdots < K_n = G.$$

Let $H < G$ be a subgp. Then we have a series

$$\{1\} = H \cap K_0 < H \cap K_1 < \cdots < H \cap K_n = K.$$

where $H \cap K_i / H \cap K_{i-1} \hookrightarrow K_i / K_{i-1}$ is abelian.

($H \cap K_i \to K_i / K_{i-1}$ with kernel $H \cap K_{i-1}$. So by 1st iso thm)

If $\bar{G}$ is a quotient gp of G and $\pi: G \to \bar{G}$ projection

Let $\bar{K}_i$ be the image of K_i in $\bar{G}$. Then

$$\{1\} = \bar{K}_0 < \bar{K}_1 < \cdots < \bar{K}_n = \bar{G}.$$

Note that $K_i/K_{i-1} \twoheadrightarrow \bar{K}_i/\bar{K}_{i-1}$ abelian

(here $K_i \to \bar{K}_i \to \bar{K}_i/\bar{K}_{i-1}$ with K_{i-1} in the kernel.

So $K_i/K_{i-1} \twoheadrightarrow \bar{K}_i/\bar{K}_{i-1}$). $\square$

Prop. Let $N \trianglelefteq G$. If N and G/N are both solvable, then G is solvable.

Pf. Let

$$\{1\} = H_0 < H_1 < \cdots < H_m = N$$

$$\{1\} = \bar{K}_0 < \bar{K}_1 < \cdots < \bar{K}_n = \bar{G} := G/N$$

be solvable series.

Let $\pi : G \to \bar{G}$ and $K_i := \pi^{-1}(\bar{K}_i)$. (So $\bar{K}_i = K_i/N$)

Then by 3rd iso thm. $\bar{K}_i/\bar{K}_{i-1} \cong K_i/K_{i-1}$.

Thus $\{1\} = H_0 < H_1 < \cdots < H_m = N = K_0 < K_1 < \cdots < K_n = G$ is a solvable series of G. $\square$

Derived series.

Recall that $[G,G] \trianglelefteq G$ is the subgp generated by

$$[a,b] := aba^{-1}b^{-1} \quad \forall a,b \in G$$

We call $[G,G]$ the 1^{st} derived subgp and denote by $G' = G^{(1)}$.

Define the 2^{nd} derived subgp $G^{(2)} = (G')'$,

3^{rd} $G^{(3)} = (G'')'$

Def. The derived series of G is

$$G > G^{(1)} > G^{(2)} > \cdots$$

eg. $G = S_3$, then $S_3 > A_3 > \{1\} > \{1\} > \cdots$

$G = S_5$, then $S_5 > A_5 > A_5 > \cdots$

Prop. G is solvable iff $G^{(k)} = \{1\}$ for some k.

Pf. $\Leftarrow$ by definition

$\Rightarrow$ Suppose $\{1\} = H_0 < H_1 < \cdots < H_n = G$ is a solvable series.

Since G/H_{n-1} is abelian, $G^{(1)} \subset H_{n-1}$

Thus $G^{(1)} H_{n-2} < H_{n-1}$. So by 2^{nd} iso thm

$$G^{(1)}/(G^{(1)} \cap H_{n-2}) \cong G^{(1)} H_{n-2}/H_{n-2} < H_{n-1}/H_{n-2} \text{ abelian}$$

So $G^{(2)} < G^{(1)} \cap H_{n-2} < H_{n-2}$.

Repeating this argument, we have $G^{(i)} < H_{n-i}$ $\forall i$.

Hence $G^{(k)} = \{1\}$ for some k $\square$